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Abstract: This work is devoted to the modeling of space charge dominated particle beams in the paraxial approximation with several types of external focusing fields (uniform, periodic and alternating gradient). A solid mathematical background for numerical beam simulation is established. The Kapchinsky-Vladimirsky distribution which can be matched exactly or numerically to the focusing channel is first studied. Then the matched KV beam is used to approximately match arbitrary beams. Moreover Water-Bag and Maxwell Boltzmann beams are studied to give analytical solution for code validation. Finally, numerical simulations are presented in different configurations.

Key-words: Kinetic equations, Vlasov equation, paraxial approximation, Charged particle beams

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Modélisation et simulation numérique de faisceaux de particules chargées intenses dans le cadre de l'approximation paraxiale

Résumé : Nous présentons dans ce travail la modélisation de faisceaux de particules intenses dans lesquels les effets dus au champ auto-consistant sont dominants. Nous rappelons la dérivation de l'approximation paraxiale à partir des équations de Vlasov-Maxwell. Nous considérons plusieurs configurations de champs appliqués (uniforme, périodique, gradient alterné). Nous établissons un cadre mathématique rigoureux pour la simulation de faisceaux de particules et expliquons comment un faiceau de particule quelconque peut être adapté au canal de focalisation en utilisant le faisceau KV équivalent. De plus nous donnons des exemples de solutions analytiques qui permettent de valider les codes. Finalement, nous présentons quelques résultats de simulations.

Mots-clés : Equations cinétiques, équation de Vlasov, approximation paraxiale, faisceaux de particules chargées.

Contents

1 Introduction

A model which can be used in many cases for the study of plasma as well as beam propagation is the Vlasov equation coupled with the Maxwell or Poisson equations to compute the self-consistent fields. It describes the evolution of a system of particles under the effects of external and self-consistent fields. The unknown $f(t, x, v)$, depending on the time t, the position x, and the velocity v, represents the distribution of particles in phase space for each species. However, the numerical solution of the Vlasov-Maxwell system in phase space requires an important computational effort. In such a situation, we have to take into account the particularities of the physical problem (characteristic length, geometric and physical characteristics) to derive approximate models leading to cheaper computations.

This paper is devoted to the analysis and numerical study of the Paraxial model, which is often used in Accelerator Physics for analysing the propagation of beams possessing an optical axis, which is assumed to be a straight line. For a physicist's derivation of this model one can refer to the recent book by Davidson and Qin [1].

In a previous study P. Degond and P.-A. Raviart [2, 12] presented a rigourous study of the paraxial model as an approximation of the steady-state Vlasov-Maxwell equations. The particles of the beam remain close to this optical axis, so that the transverse width l of the beam is very small compared to a characteristic length L, and have about the same kinetic energy. A scaling of the Vlasov-Maxwell equations is given which reflects the geometric and physical characteristics of a paraxial beam. In the scaled equation $\eta = l/L$ plays the role of a small parameter and an asymptotic expansions of the various physical quantities in powers of η are derived. Finally, the authors give a complete analysis of the linear model : existence of invariants, presentation of K-V and Water-bag distributions, which are exact solutions of the Paraxial model.

In [9], the authors also use an asymptotic expansion technique to treat the case of high energy short beams, considering a bunch of highly relativistic charged particles in the interior of a perfectly conducting hollow tube. A new paraxial model is derived using a frame which moves along the optical axis with the light velocity and the bunch is evolving slowly. We also mention the work of A. Nouri [10] on axisymmetric laminar beams.

These works are really interesting from a mathematical point of view since small parameters are clearly identified and derivations of models are rigourous. However, they do not give any information about the numerical simulation of beams obeying these models.

The aim of the present work is to do the link between the mathematical models and the numerical simulation of actual beams. In particular we describe how a specific beam called a KV beam can be matched to the external focusing fields and how the matched KV beam can be used to approximately match an arbitrary beam.

In the next section, we give a review of the Paraxial model obtained from the Vlasov-Maxwell equations. Then, we present an intuitive existence proof of a K-V solution for the Paraxial model and give some definitions useful for beam focusing in a periodic framework. We next focus our attention to the axisymmetric case, for which we give a new formulation using the invariance of the canonical momentum. This formulation is particularly well adapted for numerical computations. For this model, we give a rigourous presentation of stationary solutions in the non-linear case (K-V and Water-bag solutions). Finally, a simple method based on a time splitting scheme for the numerical resolution of the axisymmetric case is presented with different numerical tests illustrating the accuracy of the method.

2 The paraxial model

The steady-state relativistic Vlasov equation reads

$$\mathbf{v} \cdot \nabla_{\mathbf{x}} \tilde{f} + q\,(E + \mathbf{v} \times B) \cdot \nabla_{\mathbf{p}} \tilde{f} = 0, \tag{1}$$

where q is the charge and m is the mass of one particle, c the velocity of light in free space, $\mathbf{v} = \frac{\mathbf{p}}{\gamma m}$, $\beta = |\mathbf{v}|/c$, $\gamma = (1-\beta^2)^{-\frac{1}{2}}$, $\tilde{f}(\mathbf{x},\mathbf{p})$ represents the distribution function of one species of particles (ions, electrons), depending on position $\mathbf{x} \in \mathbb{R}^3$ and velocity $\mathbf{p} \in \mathbb{R}^3$. From the distribution function $\tilde{f}$, we compute the charge and current densities

$$\rho(\mathbf{x}) = q \int_{\mathbb{R}^3} \tilde{f}(\mathbf{x},\mathbf{p}) d\mathbf{p}, \quad \mathbf{j}(\mathbf{x}) = q \int_{\mathbb{R}^3} \mathbf{v}\, \tilde{f}(\mathbf{x},\mathbf{p}) d\mathbf{p} \tag{2}$$

and the electromagnetic fields $\mathbf{E}$ (electric field) and $\mathbf{B}$ (magnetic field) are given by the steady-state Maxwell equations

$$\begin{cases} \nabla \times \mathbf{B} = -\mu_0\, \mathbf{j}, \\ \nabla \times \mathbf{E} = 0, \\ \nabla \cdot \mathbf{E} = \dfrac{\rho}{\varepsilon_0}, \quad \nabla \cdot \mathbf{B} = 0. \end{cases} \tag{3}$$

The numerical solution of the full Vlasov-Maxwell system can be extremely expensive in computer time. Therefore, whenever possible, it is essential to find simplified models which approximate the Maxwell equations in some sense.

In the case of particle beams, we can derive a simplified model based on the following assumption which are often satisfied in physical problems involving particle beams.

- The beam is steady-state : All partial derivatives with respect to time vanish.
- The beam is sufficiently long so that longitudinal self-consistent forces can be neglected.
- The beam is propagating at constant velocity v_b along the propagation axis z.
- Electromagnetic self-forces are included.
- $p_x, p_y << p_b$ where $p_b = \gamma m v_b$ is the beam momentum. It follows in particular that

$$\beta \approx \beta_b = (v_b/c)^2, \qquad \gamma \approx \gamma_b = (1-\beta_b^2)^{-1/2}$$

- the beam is thin: the transverse dimensions of the beam are small compared to the characteristic longitudinal dimension.

The Paraxial model of approximation of Vlasov-Maxwell's equations is obtained by retaining only the first terms in the asymptotic expansion of the distribution function and the electromagnetic fields with respect to $\eta = l/L$, where l denotes the transverse characteristic length and L is the longitudinal characteristic length [2].

$$\eta = l/L \ll 1.$$

Moreover, for simplicity we will neglect the variation with respect to the longitudinal mean velocity v_b.

Let us now introduce

$$f = f(z, \mathbf{x}, \mathbf{v}),\ \mathbf{x} = (x, y),\ \mathbf{v} = (v_x, v_y),\ \Phi = \Phi(\mathbf{x}, z),\ \mathbf{B} = \left(B_x\left(\mathbf{x}, z\right), B_y\left(\mathbf{x}, z\right)\right)$$

where the new distribution function f is linked to the solution $\tilde{f}$ of the original Vlasov equation (1) by $\tilde{f}(x, y, z, p_x, p_y, p_z) = f(z, \mathbf{x}, \mathbf{v})\delta(p_z - p_b)$. Then making the assumptions above f is a solution (in the sense of distributions of)

$$v_b \frac{\partial f}{\partial z} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \frac{q}{\gamma_b m} \mathbf{F} \cdot \nabla_{\mathbf{v}} f = 0, \tag{4}$$

$$\mathbf{E} = -\nabla_{\mathbf{x}} \Phi, \quad -\Delta_{\mathbf{x}} \Phi = q\, n/\epsilon_0, \quad n = \int_{\mathbb{R}^2} f d\mathbf{v}, \tag{5}$$

$$\left\{ \begin{array}{rcl} \dfrac{\partial B_y}{\partial x} - \dfrac{\partial B_x}{\partial y} &=& \mu_0\, q\, v_b\, n, \\[2ex] \dfrac{\partial B_x}{\partial x} + \dfrac{\partial B_y}{\partial y} &=& -\dfrac{dB_z}{dz} \end{array} \right. \tag{6}$$

and $\mathbf{F} = (F_x, F_y)$ is given by

$$F_x = -\frac{\partial \Phi}{\partial x} - v_b\, B_y + v_y\, B_z, \quad F_y = -\frac{\partial \Phi}{\partial y} + v_b\, B_x - v_x\, B_z.$$

Since Maxwell equations are linear, we can split the transverse fields into their external and self-consistent parts :

$$\mathbf{E} = \mathbf{E}^e + \mathbf{E}^s, \quad \mathbf{B} = \mathbf{B}^e + \mathbf{B}^s.$$

which are respectively of the forms

$$\mathbf{E}^e = -\nabla_{\mathbf{x}} \Phi^e, \quad \mathbf{B}^e = -\nabla_{\mathbf{x}} \chi^e, \tag{7}$$

and

$$\mathbf{E}^s = -\nabla_{\mathbf{x}} \Phi^s, \quad \mathbf{B}^s = \mathbf{curl}_{\mathbf{x}} \psi^s = (\partial_y \psi^s, -\partial_x \psi^s). \tag{8}$$

On the one hand, from the equations (6), we check that the functions ϕ^e and χ^e satisfy

$$-\Delta_{\mathbf{x}} \Phi^e = \quad 0,\ -\Delta_{\mathbf{x}} \chi^e = \quad -\frac{dB_z}{dz}, \tag{9}$$

where B_z is an external magnetic field.

On the other hand, from the equations (5) and (6) the self-consistent forces satisfy

$$-\Delta_{\mathbf{x}}\Phi^s = \quad q\,n/\epsilon_0,\ -\Delta_{\mathbf{x}}\psi^s = \quad \mu_0\,v_b\,q\,n = q\frac{v_b}{\epsilon_0 c^2}\,n. \tag{10}$$

Then, we have

$$\psi^s = \frac{v_b}{c^2}\Phi^s$$

and the self-consistent force field is given by

$$F_x^s = \frac{q}{\gamma_b m}(-\frac{\partial\Phi^s}{\partial x} - v_b\,B_y^s) = -\frac{q}{\gamma_b m}(1-\beta_b^2)\frac{\partial\Phi^s}{\partial x} = -\frac{q}{\gamma_b^3 m}\frac{\partial\Phi^s}{\partial x}, \tag{11}$$

$$F_y^s = \frac{q}{\gamma_b m}(-\frac{\partial\Phi^s}{\partial y} + v_b\,B_x^s) = -\frac{q}{\gamma_b m}(1-\beta_b^2)\frac{\partial\Phi^s}{\partial y} = -\frac{q}{\gamma_b^3 m}\frac{\partial\Phi^s}{\partial y}, \tag{12}$$

where $\beta = v_b/c$.

For the external forces, we consider the three different types of external focusing forces which are mostly used in accelerators for modeling purposes:

1. Uniform focusing by a uniform electric field of the form

$$\mathbf{E}(\mathbf{x}) = -\frac{\gamma_b m}{q}\omega_0^2\,(x\,\mathbf{e}_x + y\,\mathbf{e}_y).$$

2. Periodic focusing by a magnetic field of the form

$$\mathbf{B}(\mathbf{x}) = B(z)\,\mathbf{e}_z - \frac{1}{2}B'(z)\,(x\,\mathbf{e}_x + y\,\mathbf{e}_y),$$

   where the longitudinal component of the magnetic field $B(z)$ is given and satisfies the periodicity condition $B(z+S) = B(z)$.

3. Alternating gradient focusing:

   - either by a magnetic field of the form

$$\mathbf{B}(\mathbf{x}) = B'(z)\,(y\,\mathbf{e}_x + x\,\mathbf{e}_y),$$

     which corresponds to a potential

$$\psi^e(\mathbf{x}) = -\frac{1}{2}B'(z)(x^2 - y^2),$$

   - or by an electric field of the form

$$\mathbf{E}(\mathbf{x}) = E'(z)\,(x\,\mathbf{e}_x - y\,\mathbf{e}_y),$$

     which corresponds to a potential

$$\Phi^e(\mathbf{x}) = -\frac{1}{2}E'(z)(x^2 - y^2).$$

The Vlasov equation (4) we consider, can be interpreted, dividing all the terms by the strictly positive velocity v_b, as a transverse Vlasov equation where z plays the role of time. We then get the paraxial model

$$\frac{\partial f}{\partial z} + \frac{\mathbf{v}}{v_b}\cdot\nabla_{\mathbf{x}} f + \frac{q}{\gamma_b m v_b}\left(-\frac{1}{\gamma_b^2}\nabla\Phi^s + \mathbf{E}^e + (\mathbf{v}, v_b)^T \times \mathbf{B}^e\right)\cdot\nabla_{\mathbf{v}} f \;=\; 0, \tag{13}$$

coupled with the Poisson equation

$$-\Delta_{\mathbf{x}}\Phi^s = \frac{q}{\epsilon_0}\int_{\mathbb{R}^2} f(z,\mathbf{x},\mathbf{v})\,d\mathbf{v}. \tag{14}$$

The characteristic curves associated to this Vlasov equation are given by

$$\left\{\begin{array}{l} x' = \dfrac{v_x}{v_b}, \\[2ex] y' = \dfrac{v_y}{v_b}, \\[2ex] v_x' = -\dfrac{q}{\gamma_b^3 m v_b}\dfrac{\partial\Phi^s}{\partial x} + \dfrac{q}{\gamma_b m v_b}\left(E_x^e + v_y\,B_z^e - v_b\,B_y^e\right) \\[2ex] v_y' = -\dfrac{q}{\gamma_b^3 m v_b}\dfrac{\partial\Phi^s}{\partial y} + \dfrac{q}{\gamma_b m v_b}\left(E_y^e - v_x\,B_z^e + v_b\,B_x^e\right), \end{array}\right. \tag{15}$$

where the notation $'$ corresponds to the derivative with respect the the longitudinal variable z.

The Paraxial model is much simpler than the full Vlasov-Maxwell model. On the one hand, one replaces the stationary Vlasov equation by the Paraxial Vlasov equation (4) where the longitudinal co-ordinates z plays the role of a time variable and which can be solved numerically by a marching procedure. On the other hand the stationary Maxwell's equations are replaced by the two dimensional Poisson equations (10) where z only acts like a parameter.

3 The Kapchinky-Vladimirsky (KV) distribution

Let us introduce the two functions of z, κ_x et κ_y which are given by the focusing forces, $N_0 = \int f\,d\mathbf{x}\,d\mathbf{v}$ the total number of particles and the parameter K which is a dimensionless parameter linked to the self-consistent force called *perveance*

$$K = \frac{q^2 N_0}{2\pi\epsilon_0\gamma_b^3 m v_b^2}.$$

The perveance is linked to the plasma frequency ω_p by the relation

$$K = \frac{\omega_p^2 a^2}{2\gamma_b^2 v_b^2}.$$

ϵ_x, ϵ_y are parameters, called *emittance*, linked to the constant areas of ellipses in the planes $x - x'$ et $y - y'$. Moreover, we will assume that $\kappa_x(z)$ and $\kappa_y(z)$ satisfy

$$\frac{d}{dz}\kappa_x(z) \leq C\,\kappa_x(z), \quad \frac{d}{dz}\kappa_y(z) \leq C\,\kappa_y(z) \tag{16}$$

Theorem 1 *Let us define the K-V distribution [7]*

$$f(z,x,y,P_x,P_y) = \frac{N_0}{\pi^2\,\epsilon_x\,\epsilon_y}\delta_0\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{(a\,P_x - a^{'}\,x)^2}{\epsilon_x^2}+\frac{(b\,P_y - b^{'}\,y)^2}{\epsilon_y^2}-1\right), \tag{17}$$

where a, b are the solutions of the so-called envelope equations (see [1, 13])

$$a^{''} + \kappa_x(z)\,a - \frac{2K}{a+b} - \frac{\epsilon_x^2}{a^3} = 0, \quad b^{''} + \kappa_y(z)\,b - \frac{2K}{a+b} - \frac{\epsilon_x^2}{b^3} = 0, \tag{18}$$

where $a^{'}$ denotes the derivative with respect to z. Then, f is a measure solution of the paraxial model (13)-(14) and the self-consistant electric field solution of the Poisson equation satisfies

$$E^s(x,y) = \begin{cases} \dfrac{qN_0}{\pi\epsilon_0(a+b)}\begin{pmatrix} x/a \\ y/b \end{pmatrix} & \text{if } \frac{x^2}{a^2}+\frac{y^2}{b^2} \leq 1, \\ \to 0 & \text{as } x,y \to \infty. \end{cases} \tag{19}$$

Let us first give a sense to the distribution f , which can be understood thanks to the Co-area formula (see [3]). Indeed, define $H_z = H_z(x,y,P_x,P_y)$ by

$$H_z(x,y,P_x,P_y) = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{(a\,P_x - a^{'}\,x)^2}{\epsilon_x^2}+\frac{(b\,P_y - b^{'}\,y)^2}{\epsilon_y^2}.$$

Then for any $e \in H_z(\mathbb{R}^4)$, the manifold

$$H_z^{-1}(e) = \left\{(x,y,P_x,P_y) \in \mathbb{R}^4; e = H_z(x,y,P_x,P_y)\right\}.$$

is an hyper-surface of $\mathbb{R}^4$. Denote by $dS(x,y,P_x,P_y)$ its Euclidian surface element and by $N_z(e)$ the density of states e :

$$N_z(e) = \int_{k\in H_z^{-1}(e)} \frac{dS(k)}{\nabla_k H_z(k)}.$$

The distribution f given by (30) is then a measure of $\mathbb{R}^4$ defined as follows : for any Borel set B of $\mathbb{R}^4$

$$f(z)(B) = \int_{k\in B} \frac{dS(k)}{\nabla_k H_z(k)}. \tag{20}$$

Before proving this Theorem, let us give the main arguments. We will first compute some invariants of the Vlasov equation (13) when the total electromagnetic force is linear with respect to (x, y). Then, we will prove that the K-V distribution is a measure solution of the Vlasov equation. Finally, we will consider the coupling of the Vlasov and Poisson equations to conclude the proof.

Let us first consider the Vlasov equation with a total force field linear with respect to (x, y), the characteristic curves are then given by the Mathieu-Hill differential equations

$$x^{'} = P_x, \quad P_x^{'} = -\overline{\kappa}_x(z)\, x, \quad y^{'} = P_y, \quad P_y^{'} = -\overline{\kappa}_y(z)\, y, \tag{21}$$

where $\overline{\kappa}_x(z)$ and $\overline{\kappa}_y(z)$ characterize the total (self-consitent + applied) linear force field. Many systems, such as quadrupole channels, have two planes of symmetry, where the forces may differ. However, as long as there is no coupling between these two forces, the theory is the same, and in the following, we will only consider the motion in the x-direction.

Proposition 1 (x, P_x) *is a solution of (21) if and only if it satisfies*

$$\frac{x^2}{a^2} + \frac{(a\, P_x - a^{'}\, x)^2}{\epsilon_x^2} = 1, \tag{22}$$

where $a = \sqrt{\epsilon_x}\, w_x$ *and*

$$w_x^{''} + \overline{\kappa}_x(z)\, w_x - \frac{1}{w_x^3} = 0 \tag{23}$$

where $w_x(z)$ is a function of z.

Proof. Let us consider the equation of motion for one particle in the linear force field:

$$x^{'} = P_x, \quad P_x^{'} = -\overline{\kappa}_x(z)\, x. \tag{24}$$

The generalized solutions of (24) can be expressed as

$$u(z) = w_x(z)\, \exp(+i\, \Psi_x(z)), \quad v(z) = w_x(z)\, \exp(-i\, \Psi_x(z)). \tag{25}$$

Then, plugging the solution (25) into equation (24)

$$w_x^{''} + 2\, i\, w_x^{'}\, \Psi_x^{'} + \left(\overline{\kappa}_x(z) + i\, \Psi_x^{''} - (\Psi_x^{'})^2\right)\, w_x = 0,$$

and the real and imaginary parts of this last term respectively satisfy the following relations

$$w_x^{''} + \left(\overline{\kappa}_x(z) - (\Psi_x^{'})^2\right)\, w_x = 0, \tag{26}$$

$$2\, w_x^{'}\, \Psi_x^{'} + w_x\, \Psi_x^{''} = 0. \tag{27}$$

From relation (27), we get an expression for the function Ψ_x

$$\Psi_x^{'} = C/w_x^2, \tag{28}$$

where the constant C only depends on the initial data. Let us set C to one to define w_x and reintroduce the dependancy on the initial data through a constant ϵ_x. Although w_x is not explicit known, it satisfies the following differential equation, which is obtained from (26) and (28)

$$w_x^{''} + \overline{\kappa}_x(z)\, w_x - \frac{1}{w_x^3} = 0.$$

Any real solution of (24) can be written as a linear combination of u and v in the following form

$$x(z) = \sqrt{\epsilon_x}\, w_x(z)\, \cos(\Psi_x(z) + \phi_0),$$

where ϵ_x and ϕ_0 only depend on the initial data. The derivative is then given by

$$x^{'}(z) = \sqrt{\epsilon_x}\left(w_x^{'}(z)\, \cos(\Psi_x(z) + \phi_0) - \frac{1}{w_x(z)}\, \sin(\Psi_x(z) + \phi_0)\right).$$

Moreover,

$$\begin{aligned}(w_x\, x^{'})^2 &= (w_x^{'})^2\, x^2 + \epsilon_x\, \sin^2(\Psi_x + \phi_0) - 2\, w_x^{'}\, x\, \sqrt{\epsilon_x}\, \sin(\Psi_x + \phi_0) \\ &= (w_x^{'})^2\, x^2 + \epsilon_x - \frac{x^2}{w_x^2} - 2\, w_x^{'}\, x\left(w_x^{'}\, x - w_x\, x^{'}\right).\end{aligned}$$

Then, introducing the so-called Twiss parameters (see [1, 13])

$$\beta_x = w_x^2, \quad \alpha_x = -w_x\, w_x^{'}, \quad \gamma_x = (1+\alpha_x^2)/\beta_x = 1/w_x^2 + (w_x^{'})^2,$$

we get that all particles with the same initial ϵ_x, but a different ψ_0 will lie on the same phase space ellipse determined by α_x, β_x and γ_x

$$\gamma_x\, x^2 + 2\, \alpha_x\, x\, P_x + \beta_x\, P_x^2 = \epsilon_x. \tag{29}$$

Finally, we set $a = \sqrt{\epsilon_x}\, w_x$. Then, we get an invariant of the Vlasov equation in the following form

$$\frac{x^2}{a^2} + \frac{(a\, P_x - a^{'}\, x)^2}{\epsilon_x^2} = 1.$$

The existence and uniqueness of a smooth solution local in time come from the smoothness of the focusing function. Moreover from the following estimate

$$\frac{d}{dz}\left((w_x^{'})^2 + \overline{\kappa}_x(z)\, w_x^2 + \frac{1}{w_x^2}\right) = \frac{d}{dz}\left(\overline{\kappa}_x(z)\right)\, w_x^2 \le C\, \kappa_x(z)\, w_x^2$$

we conclude that the solution is global in time. □

Proposition 1 implies that (22) is an invariant of the Vlasov equation for which the characteristic curves yield the Mathieu-Hill equations (21).Then, $N_0, \epsilon_x, \epsilon_y$ being given, let us consider the K-V distribution (17)

$$f(z, x, y, P_x, P_y) = \frac{N_0}{\pi^2\, \epsilon_x\, \epsilon_y} \delta_0 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(a\, P_x - a^{'}\, x)^2}{\epsilon_x^2} + \frac{(b\, P_y - b^{'}\, y)^2}{\epsilon_y^2} - 1 \right), \qquad (30)$$

where $a = \sqrt{\epsilon_x}\, w_x$, $b = \sqrt{\epsilon_y}\, w_y$ and w_x and w_y are solutions of equation of type (23).

Proposition 2 *The K-V distribution given by (17) is a measure solution of the Vlasov equation defined by the characteristic curves (21).*

Before giving the proof of Proposition 2, let us recall the definition of a measure solution of the Vlasov equation [11].

Consider the transport equation

$$\frac{\partial u}{\partial t} + \mathrm{div}_x(a(t, x) u(t, x)) = 0, \qquad (31)$$

where $a \in L^1_{loc}(\mathbb{R}^+; C^1(\mathbb{R}^n))^n$. For every $u_0 \in \mathcal{M}^1(\mathbb{R}^n)$, the solution of (31) is given by : for any Borel set B,

$$u(t)(B) = u_0(x^{-1}(t)(B))$$

where the continuous map $x(t)$ is solution of

$$\frac{dx}{dt} = a(t, x(t)), \quad x(0) = x.$$

Proof of Proposition 2. Let us denote by $\mathbf{x}(z) = (x(z), y(z), P_x(z), P_y(z))$ the characteristic curves solution of

$$\begin{array}{ll} x^{'} = P_x, & P_x^{'} = -\overline{\kappa}_x(z)\, x, \\ y^{'} = P_y, & P_y^{'} = -\overline{\kappa}_y(z)\, y. \end{array}$$

From the definition of the K-V distribution given by (20), the result becomes straightforward. Indeed, for any Borel set B of $\mathbb{R}^4$

$$\begin{aligned} f(z)(B) &= \int_{k \in B} \frac{dS(k)}{\nabla_k H_z(k)} = \int_{k \in B} \frac{dS(k)}{\nabla_k H_0(\mathbf{x}^{-1}(z, k)} \\ &= \int_{k' \in \mathbf{x}^{-1}(z)(B)} \frac{dS(k')}{\nabla_{k'} H(0, k')} \\ &= f_0\left(\mathbf{x}^{-1}(z)(B)\right) \end{aligned}$$

$\square$

Now, we will consider the coupling of the Vlasov equation with the Poisson equation, in order to take into account self-consistent electromagnetic fields.

Proposition 3 *Let us consider the K-V distribution (17). Then, the space charge density is given by*

$$q\,n(x,y) = \begin{cases} \dfrac{q\,N_0}{\pi\,a\,b} & \textit{if } \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1, \\ 0 & \textit{else.} \end{cases} \tag{32}$$

and the self-consistent electric field solution of the Poisson equation is

$$E^s(x,y) = \begin{cases} \dfrac{q\,N_0}{\pi\,\epsilon_0\,(a+b)} \begin{pmatrix} x/a \\ y/b \end{pmatrix} & \textit{if } \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1, \\ \to 0 & \textit{as } x, y \to \infty. \end{cases}$$

Proof. Let us first compute the space charge density, which is obtained by integrating the K-V distribution with respect to (P_x, P_y)

$$q\,n(x,y) = \frac{qN_0}{\pi^2\,\epsilon_x\,\epsilon_y} \int_{\mathbb{R}^2} \delta_0 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(a\,P_x - a^{'}\,x)^2}{\epsilon_x^2} + \frac{(b\,P_y - b^{'}\,y)^2}{\epsilon_y^2} - 1 \right) dP_x\,dP_y.$$

In fact this "integral" can be restricted to the following domain

$$S(x,y,z) = \left\{ (P_x, P_y) \in \mathbb{R}^2 \,:\, \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(a\,P_x - a^{'}\,x)^2}{\epsilon_x^2} + \frac{(b\,P_y - b^{'}\,y)^2}{\epsilon_y^2} \leq 1 \right\}.$$

Then, setting

$$\overline{P}_x = \frac{a\,P_x - a^{'}\,x}{\epsilon_x}, \quad \overline{P}_y = \frac{b\,P_y - b^{'}\,y}{\epsilon_y},$$

we get

$$q\,n(x,y) = \frac{qN_0}{\pi^2\,a\,b} \int_{\overline{S}} \delta_0 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \overline{P}_x^2 + \overline{P}_y^2 - 1 \right) d\overline{P}_x\,d\overline{P}_y,$$

where

$$\overline{S} = \left\{ (\overline{P}_x, \overline{P}_y) \in \mathbb{R}^2 \,:\, \frac{x^2}{a^2} + \frac{y^2}{b^2} + \overline{P}_x^2 + \overline{P}_y^2 \leq 1 \right\}.$$

Making the change of variables $\overline{P}_x = \sqrt{R}\cos\theta$, $\overline{P}_y = \sqrt{R}\sin\theta$ of jacobian $\frac{1}{2}$ and using the invariance by rotation

$$\begin{aligned} q\,n(x,y) &= \frac{qN_0}{\pi\,a\,b}\int_{\mathbb{R}+}\delta_0\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+R-1\right)dR, \\ &= \begin{cases} \dfrac{q\,N_0}{\pi\,a\,b} & \text{if } \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1, \\ 0 & \text{else.} \end{cases} \end{aligned}$$

Finally, since the space charge is uniform inside the ellipse, we can explicitly compute the electric field. Indeed the electric field $\mathbf{E}^s$ satisfies $\operatorname{curl}\mathbf{E}^s = 0$ and

$$\operatorname{div}\mathbf{E}^s = \frac{qN_0}{\pi\epsilon_0(a+b)}(\frac{1}{a}+\frac{1}{b}) = \frac{qN_0}{\epsilon_0\pi ab},$$

which yields the electric field

$$E^s(x,y) = \begin{cases} \dfrac{q\,N_0}{\pi\,\epsilon_0\,(a+b)}\begin{pmatrix} x/a \\ y/b \end{pmatrix} & \text{if } \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1, \\ \to 0 & \text{as } x,y\to\infty. \end{cases}$$

$\square$

From the above expression of the electric field, it follows that the focusing force F_x^s given by (11) becomes

$$F_x^s = \frac{q}{\gamma_b^3 m}\frac{qN_0}{\pi\epsilon_0(a+b)}\frac{x}{a} = \frac{2Kv_b^2}{(a+b)a}\,x,$$

and in the same way F_y^s given by (12) becomes

$$F_y^s = \frac{q}{\gamma_b^3 m}\frac{qN_0}{\pi\epsilon_0(a+b)}\frac{y}{b} = \frac{2Kv_b^2}{(a+b)b}\,y.$$

Now, we can easily prove the Theorem 1

Proof of Theorem 1. Consider a linear force field of the following form

$$\overline{\kappa}_x(z) = \kappa_x(z) - \frac{2K}{a\,(a+b)} \quad \overline{\kappa}_y(z) = \kappa_y(z) - \frac{2K}{b\,(a+b)},$$

where κ_x, and κ_y determine the external focusing force and $K = q^2\,N_0/(2\,m\,\pi\,\epsilon_0\,v_b^2)$ the self force. Since the force field is linear we can apply Proposition 1, which gives an invariant of the Vlasov equation under the condition that a, b are solution of (18)

$$a'' + \kappa_x(z)\,a - \frac{2K}{a+b} - \frac{\epsilon_x^2}{a^3} = 0, \quad b'' + \kappa_y(z)\,b - \frac{2K}{a+b} - \frac{\epsilon_y^2}{b^3} = 0,$$

This system of ODE's has an unique local solution since $\kappa_x(z)$ and $\kappa_y(z)$ are smooth enough. Moreover, from the following estimates

$$\frac{d}{dz}\left((a^{'})^2 + (b^{'})^2 + \kappa_x(z)\,a^2 + \kappa_y(z)\,b^2) + 2K\log(a+b) \,+ \frac{\epsilon_x^2}{a^2} + \frac{\epsilon_y^2}{b^2}\right)$$
$$\leq C\,(\kappa_x(z)\,a^2 + \kappa_y(z)\,b^2).$$

we can deduce that the functions a and b are non-negative and bounded. Thus, the solution is global in time.

Applying Proposition 2, we prove that the K-V distribution function f given by (17) is a solution of the Vlasov equation (15). Finally, from Proposition 3, we can check that the space charge density is uniform inside the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$$

and the self-consistent electric field solution of the Poisson equation is linear with respect to (x, y) inside this ellipse, which concludes the proof. □

4 Envelope equations

Let us now derive explicitly the envelope equations in the three focusing configurations we are interested in.

1. Uniform focusing. The focusing field is given by

$$\mathbf{E}^e(\mathbf{x}) = -\frac{\gamma_b m}{q}\omega_0^2\,(x\,\mathbf{e}_x + y\,\mathbf{e}_y).$$

   Using the characteristics equations and the expression of the self-consistent field we get

$$x'' = \frac{v_x'}{v_b} = \frac{q}{\gamma_b^3 m v_b^2}E_x^s + \frac{q}{\gamma_b m v_b^2}E_x^e = \frac{2K}{(a+b)a}x - \frac{\omega_0^2}{v_b^2}x,$$

   and

$$y'' = \frac{2K}{(a+b)a}x - \frac{\omega_0^2}{v_b^2}y.$$

   We thus obtain a Matthieu-Hill equation from which we deduce the envelope equations

$$a'' + k_0^2 a - \frac{2K}{a+b} = \frac{\epsilon_x}{a^3}, \quad b'' + k_0^2 b - \frac{2K}{a+b} = \frac{\epsilon_y}{b^3},$$

where we denote by $k_0 = \omega_0/v_b$. As k_0 et K do not depend on z, these equations admit solutions such that $a' = a'' = b' = b'' = 0$ and $a = b$ if $\epsilon_x = \epsilon_y$. Looking for such a solution, we get

$$k_0^2 a^4 - K a^2 - \epsilon_x^2 = 0.$$

Taking the positive root of this equation, we obtain

$$a = \sqrt{\frac{K + \sqrt{K^2 + 4k_0^2\epsilon_x^2}}{2k_0^2}}. \tag{33}$$

2. Periodic focusing by a magnetic field of the form

$$\mathbf{B}^e(\mathbf{x}) = B(z)\,\mathbf{e}_z - \frac{1}{2}B'(z)\,(x\,\mathbf{e}_x + y\,\mathbf{e}_y),$$

where B is S-periodic. Proceeding as in the previous case and plugging in the expression of $\mathbf{B}^e$, we get for an axisymmetric beam such that $a = b$

$$x'' = \frac{K}{a^2}x + \frac{q}{\gamma_b m v_b^2}(v_y B_z + v_b \frac{B'}{2} y), \quad y'' = \frac{K}{a^2}x - \frac{q}{\gamma_b m v_b^2}(v_x B_z + v_b \frac{B'}{2} x).$$

We then introduce the relativistic Larmor frequency divided by the beam velocity

$$\omega_L(z) = \frac{qB(z)}{2\gamma_b m v_b}.$$

It follows

$$x'' = \frac{K}{a^2}x + 2\omega_L y' + \omega_L' y, \quad y'' = \frac{K}{a^2}y - 2\omega_L x' - \omega_L' x. \tag{34}$$

These equations take a simpler form in the Larmor frame which is rotating at the Larmor frequency. Taking for θ_L a primitive of $-\omega_L$, the co-ordinates (X, Y) in the Larmor frame are obtained by a rotation of angle θ_L from co-ordinates (x, y) in the laboratory frame. We then have

$$x(z) = X(z)\cos\theta_L(z) - Y(z)\sin\theta_L(z), \quad y(z) = X(z)\sin\theta_L(z) + Y(z)\cos\theta_L(z).$$

Taking the derivative with respect to z, we get

$$\begin{aligned} x' &= (X' + \omega_L Y)\cos\theta_L(z) + (Y' - \omega_L X)\sin\theta_L(z), \\ y' &= (X' + \omega_L Y)\sin\theta_L(z) + (Y' - \omega_L X)\cos\theta_L(z), \end{aligned}$$

and

$$x'' = (X'' + 2\omega_L Y' + \omega_L' Y - \omega_L^2 X)\cos\theta_L(z) + (-Y'' + 2\omega_L X' + \omega_L' X + \omega_L^2 X)\sin\theta_L(z),$$
$$y'' = (X'' + 2\omega_L Y' + \omega_L' Y - \omega_L^2 X)\sin\theta_L(z) - (-Y'' + 2\omega_L X' + \omega_L' X + \omega_L^2 Y)\cos\theta_L(z).$$

Plugging these expressions into (34), we obtain the particle motion equations in the Larmor frame

$$X'' - \frac{K}{a^2}X + \omega_L^2 X = 0, \quad Y'' - \frac{K}{a^2}Y + \omega_L^2 Y = 0.$$

The equations for X and Y are both Matthieu-Hill equations

$$\bar{\kappa}_x(z) = \bar{\kappa}_y(z) = \omega_L(z)^2 - \frac{K}{a^2}.$$

The envelope equation thus reads

$$a'' + \omega_L^2 a - \frac{K}{a} = \frac{\epsilon^2}{a^3}. \tag{35}$$

The radius a of a matched beam for given K, ϵ and ω_L given can be computed by numerically solving for a periodic solution of this equation.

3. Alternating gradient focusing: in the case of a magnetic field of the form

$$\mathbf{B}^e(\mathbf{x}) = B'(z)\,(y\,\mathbf{e}_x + x\,\mathbf{e}_y),$$

the particles motion equations read:

$$x'' = \frac{2K}{(a+b)a}x - \frac{qB'}{\gamma_b m v_b^2}(v_b x), \quad y'' = \frac{2K}{(a+b)a}x + \frac{qB'}{\gamma_b m v_b^2}(v_b y).$$

we thus have Matthieu-Hill equations with

$$\bar{\kappa}_x(z) = -\frac{qB'}{\gamma_b m v_b} - \frac{K}{a^2}$$

and

$$\bar{\kappa}_y(z) = \frac{qB'}{\gamma_b m v_b} - \frac{K}{a^2}.$$

Whence the envelope equations

$$a'' - \frac{q\,B'}{\gamma_b m v_b}a - \frac{2K}{a+b} = \frac{\epsilon_x^2}{a^3}, \quad b'' + \frac{q\,B'}{\gamma_b m v_b}b - \frac{2K}{a+b} = \frac{\epsilon_x^2}{b^3}. \tag{36}$$

similarly for an electric field of the form

$$\mathbf{E}(\mathbf{x}) = E'(z)\,(x\,\mathbf{e}_x - y\,\mathbf{e}_y),$$

the particles motion equations read:

$$x'' = \frac{2K}{(a+b)a}x + \frac{qE'}{\gamma_b m v_b^2}(x), \quad y'' = \frac{2K}{(a+b)a}x - \frac{qE'}{\gamma_b m v_b^2}(y). \tag{37}$$

we thus have Matthieu-Hill equations with

$$\bar{\kappa}_x(z) = \frac{qE'}{\gamma_b m v_b^2} - \frac{K}{a^2}$$

and

$$\bar{\kappa}_y(z) = -\frac{qE'}{\gamma_b m v_b^2} - \frac{K}{a^2}.$$

Whence the envelope equations

$$a'' + \frac{qE'}{\gamma_b m v_b^2} a - \frac{2K}{a+b} = \frac{\epsilon_x^2}{a^3}, \quad b'' - \frac{qE'}{\gamma_b m v_b^2} b - \frac{2K}{a+b} = \frac{\epsilon_x^2}{b^3}.$$

5 Focusing a beam given by its initial distribution function

A general beam given by an analytical expression can be focused thanks to the concept of equivalent beams introduced by Sacherer [14] and Lapostolle [8].

We shall call equivalent two beams of identical particles defined by their distribution function f if they have the same energy, the same total number of particles and the same order 2 moments.

We shall use this concept to identify a general beam with a matched KV beam. Hence a general beam will be focused by focusing its equivalent KV beam. This is performed through the following steps:

- In our model the beam velocity v_b and γ_b are uniquely determined by its energy.
- The beam current determines the total number of particles by the relation $N_0 = \frac{I}{qv_b}$.
- We can then compute the beam perveance

$$K = \frac{q^2 N_0}{2\pi\epsilon_0 \gamma_b^3 m v_b^2}.$$

- We set the focusing field from the three types we consider (uniform, periodic, alternating gradient).
- These parameters being set we look numerically for a S-periodic solution of the corresponding envelope equation where S is the period of the focusing lattice.

Thus the matched KV beam is determined.

We now consider a beam whose shape is given by a known analytical expression f_0. The matched distribution function can be computed by a scaling procedure as follows. Set

$$f(x, y, x', y') = N_0 f_0(\frac{x}{a}, \frac{y}{b}, \frac{x'}{c}, \frac{y'}{d}). \quad (38)$$

The beam velocity is chosen identical to this of the matched KV beam and so is the total number of particles N_0. So it remains to set the parameters a, b, c and d that are chosen such that x_{rms}, y_{rms}, x'_{rms} et y'_{rms} associated to the density f are the same as those of the equivalent matched KV beam. Where we define for a function $\chi(x, y, x', y')$

$$\chi_{rms}(f) = \sqrt{\frac{\int \chi(x, y, x', y')^2 f(x, y, x', y')\, dx\, dy\, dx'\, dy'}{\int f(x, y, x', y')\, dx\, dy\, dx'\, dy'}}.$$

A change of variables in the above integrals gives the relations between the rms quantities associated to f and to f_0 verifying (38) :

$$x_{rms}(f) = a x_{rms}(f_0),\ y_{rms}(f) = b y_{rms}(f_0),$$

and

$$x'_{rms}(f) = c x'_{rms}(f_0),\ y'_{rms}(f) = d y'_{rms}(f_0).$$

Using the expression of the KV distribution KV given by (17), we obtain

$$x_{rms}(f_{KV}) = \frac{a_0}{2}, \quad y_{rms}(f_{KV}) = \frac{b_0}{2},$$
$$x'_{rms}(f_{KV}) = \frac{\epsilon_x}{2a_0}, \quad y'_{rms}(f_{KV}) = \frac{\epsilon_y}{2b_0},$$

where $(a_0, 0)$ and $(b_0, 0)$ are the initial conditions corresponding to a periodic solution (of period S) of the envelope equation. These values a_0 and b_0 can be computed analytically using (33) in the case of uniform focusing and numerically in the other cases.

It now remains to identify the rms quantities associated to f and to f_{KV} to determine the parameters a, b, c and d. It follows that

$$a = \frac{a_0}{2x_{rms}(f_0)}, \quad b = \frac{b_0}{2y_{rms}(f_0)},$$
$$c = \frac{\epsilon_x}{2a_0 x'_{rms}(f_0)}, \quad d = \frac{\epsilon_y}{2b_0 y'_{rms}(f_0)}.$$

Example : let us explicit this procedure in the case when f_0 is a normalized semi-Gaussian beam of the form

$$f_0(x, y, x', y') = \frac{1}{2\pi} e^{-\frac{x'^2 - y'^2}{2}} \qquad \text{if } x^2 + y^2 < 1$$

and 0 else. In this case we have

$$x_{rms}(f_0) = y_{rms}(f_0) = \frac{1}{2}, \quad x'_{rms}(f_0) = y'_{rms}(f_0) = 1,$$

so that

$$a = a_0, \quad b = b_0, \quad c = \frac{\epsilon_x}{2a_0}, \quad d = \frac{\epsilon_y}{2b_0}.$$

Remark : in the case of an axisymmetric beam, we have $x = r\cos\theta$, $y = r\sin\theta$, the computation of x_{rms} and of y_{rms} yields

$$x_{rms} = y_{rms} = \frac{r_{rms}}{\sqrt{2}}.$$

6 The axisymmetric Vlasov equation

Now, we will restrict ourselves to the axisymmetric case, which plays an important role in applications. Assuming that the distribution function f and the external and self-consistent fields are invariant by rotation, *i.e.* independent of θ.

Another invariant of motion in the Larmor frame for an axisymmetric beam is

$$\Omega = x\,P_y - y\,P_x.$$

Let us pass to cylindrical co-ordinate,

$$r^2 = x^2 + y^2,$$

and (v_r, v_θ) will represent the cylindrical co-ordinates of the canonical momentum in the Larmor frame

$$\begin{pmatrix} v_r \\ v_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} P_x \\ P_y \end{pmatrix}.$$

Moreover, the self-consistent electromagnetic fields solutions of (9) are given by

$$\phi^e = 0, \quad \chi^e = \frac{dB_z}{dz} r^2/2,$$

so that the transverse external fields are

$$\mathbf{E}^e = 0, \quad \mathbf{B}^e = -\frac{1}{2}\frac{dB_z}{dz}\,r \tag{39}$$

Moreover, the self consistent fields in cylindrical co-ordinates satisfy

$$E_r^s = \frac{q}{\epsilon_0}\left(\frac{1}{r}\int_0^r n(z,s)\,s\,ds\right) \tag{40}$$

Finally, if we pass in cylindrical coordinates (r,v_r,v_θ), the Paraxial Vlasov equation becomes

$$v_b\frac{\partial f}{\partial z} + v_r\frac{\partial f}{\partial r} + \left(\frac{q}{m}F_r + \frac{v_\theta^2}{r}\right)\frac{\partial f}{\partial v_r} + \left(\frac{q}{m}F_\theta + \frac{v_\theta\,v_r}{r}\right)\frac{\partial f}{\partial v_\theta} = 0, \tag{41}$$

where $F_k = F_k^e + F_k^s$, $k \in \{r, \theta\}$, the external electromagnetic fields are given by

$$F_r^e = -\kappa_0(z)\,r, \quad F_\theta^e = 0, \tag{42}$$

and the self-consistent forces satisfy

$$F_r^s = \frac{q}{\epsilon_0}(1-\beta^2)\frac{1}{r}\int_0^r n(z,s)sds, \quad F_\theta^s = 0. \tag{43}$$

6.1 Construction of steady-state solutions

This section is devoted to the construction of solutions of the paraxial model (41) independent of z taking into account self-consistent forces. To this aim, we consider a uniform applied potential

$$\Phi^e(r) = k_0^2\, r^2/2.$$

We use the invariance of the transverse energy

$$H_\perp(r, v_r, v_\theta) = \frac{1}{2}\left(v_r^2 + v_\theta^2\right) + \frac{1}{2}k_0^2\, r^2 + \frac{q}{m}(1-\beta^2)\Phi^s(r).$$

Considering a distribution function of the form $f(H_\perp)$, it is easy to check that it is a solution of the Paraxial model (41), but the main difficulty is the determination of the charge density $n(r)$ and the self-consistent potential $\Phi^s(r)$. Assume f is compactly supported, then the space charge density n is given by

$$\begin{aligned} n(r) &= \int_{\mathbb{R}^2} f(H_\perp(r, v_r, v_\theta))dv_r\, dv_\theta = \int_0^{2\,\pi}\int_0^{+\infty} f(\overline{H}_\perp(r, v, \psi))v dv d\psi, \\ &= \pi\int_0^{+\infty} f(\overline{H}_\perp(r, v^2))dv^2, \end{aligned}$$

where $\overline{H}_\perp(r, v, \psi) = \overline{H}_\perp(r, v)$ is the expression of $H_\perp$ in polar velocity co-ordinates.

The space charge potential Φ^s is then found by solving the Poisson equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Phi^s}{dr}\right) = -\frac{q\,\pi}{\epsilon_0}\int_0^{+\infty} f(\overline{H}_\perp)(r, v^2)dv^2.$$

Now let us denote by $W(r)$ the potential energy

$$W(r) = \frac{1}{2}k_0^2\, r^2 + \frac{q}{m}(1-\beta^2)\Phi^s(r). \tag{44}$$

We first observe that the potential energy $W(r)$ is positive. Indeed, from the Poisson equation, we check that $d\Phi^s/dr$ is negative and the potential is vanishing when r goes to infinity, which means that the self-consistent potential is positive decreasing.

Let us set H_0 the maximum transverse energy, which corresponds to the upper bound of the support of f. We introduce the radius $a \in [0, +\infty]$ for which the potential energy $W(r)$ is maximal

$$H_0 = W(a).$$

Note that since the applied potential goes to infinity for large r, when the distribution function f is compactly supported i.e. H_0 is finite then the radius a is also finite. From these notations, we introduce the maximal kinetic energy $b^2(r)/2$ for any $r \geq 0$, which is given by

$$W(a) = H_0 = \frac{1}{2}b^2(r) + W(r),$$

or

$$\frac{1}{2}b^2(r) = W(a) - W(r) \geq 0,$$

and the space charge density can be re-written in the following form

$$n(r) = \pi \int_0^{+\infty} f(\overline{H_\perp}(r, v^2))dv^2 \; = \; \pi \int_0^{W(a)-W(r)} f(v^2(r))dv^2(r) = \pi \int_{W(r)}^{W(a)} f(H_\perp)dH_\perp.$$

The K-V distribution. If we consider the following distribution

$$f(s) = \delta_0(s - H_0),$$

we recover a simple K-V distribution for the axisymmetric case.

The Waterbag distribution. Let us treat the case where the distribution function is uniform

$$f(s) = \begin{cases} f_0 & \text{if } 0 \leq s \leq H_0, \\ 0 & \text{else} \end{cases}$$

The space charge density is

$$q\, n(r) = 2\pi\, q f_0\, (W(a) - W(r))$$

and the Poisson equation becomes

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Phi^s}{dr}\right) = -\frac{q\, 2\pi\, f_0}{\epsilon_0}\,(W(a) - W(r))\,. \tag{45}$$

If we set

$$k_1^2 = \frac{2\,\pi\, q\, f_0}{\epsilon_0}\frac{q}{m}(1 - \beta^2),$$

and

$$U(r) = W(r) - W(a) + 2\,\frac{k_0^2}{k_1^2},$$

then, we can replace (45) by the following equation for $U(r)$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dU}{dr}\right) - k_1^2 U(r) = 0.$$

As we will see later the parameter k_1 plays an important role in the definition of a Waterbag distribution, since it allows to determine the shape of the beam. Thus, setting $y(x) = U(r)$ where $x = k_1\, r$, the function $y(x)$ satisfies the first Bessel differential equation

$$x\, y'' + y' = x\, y, \tag{46}$$

and $y = I_0(x)$, where I_0 is the modified first Bessel function. Finally,

$$U(r) = C\, I_0(k_1\, r).$$

Indeed, the solution of (46) is defined up to a multiplicative constant. However, we know that $U(a) = 2\, k_0^2/k_1^2$, then

$$U(r) = \frac{2\, k_0^2}{k_1^2}\, \frac{I_0(k_1\, r)}{I_0(k_1\, a)}.$$

Moreover, $U(r)$ is an increasing function of r, which means that the potential energy $W(r)$ is also increasing and positive. Then, the maximal potential energy is obtained at the maximal radius of the beam r_{max} and we have $a = r_{max}$. We also recall that for any $0 < r \leq a$, the maximal kinetic energy is given by

$$\frac{1}{2} b^2(r) = W(a) - W(r),$$

and since $W(r)$ is increasing, the velocity $a' = b(0)$ corresponds to the velocity of the maximal kinetic energy. Since the kinetic energy is an increasing function with respect to $v = (v_r^2 + v_\theta^2)^{1/2}$, the point a' corresponds to the maximal modulus of the distribution function f. Finally the boundary of the particle beam in phase space is given by

$$H_0 = \frac{1}{2}\left(v_r^2 + v_\theta^2\right) + W(r) = W(a),$$

or

$$H_0 = \frac{a'^2}{2} + W(0) = W(a).$$

Finally, we get the following relation

$$\frac{v_r^2 + v_\theta^2}{a'^2} + \frac{W(r) - W(0)}{W(a) - W(0)} = 1,$$

where $W(r)$ satisfies

$$W(r) = W(a) - 2\frac{k_0^2}{k_1^2}\left(1 - \frac{I_0(k_1\, r)}{I_0(k_1\, a)}\right).$$

Then, the phase space contours of the distribution function are described by the following relation

$$\frac{v_r^2 + v_\theta^2}{a'^2} + \frac{I_0(k_1\, r) - 1}{I_0(k_1\, a) - 1} = 1, \tag{47}$$

and the charge density is given by

$$n(r) = 4\,\pi f_0 \frac{k_0^2}{k_1^2}\left(1 - \frac{I_0(k_1\, r)}{I_0(k_1\, a)}\right).$$

The charge densities obtained for different values of $k_1\,a$ are displayed in Figure 1. Usually physicists know the macroscopic values of the beam, but not the value of f_0. For instance, assume the the total current of the beam is known

$$I = q\,v_b \int_0^a n(r) r dr = 4\,\pi\,q\,v_b f_0 \frac{k_0^2}{k_1^2} \int_0^a (1 - \frac{I_0(k_1\,r)}{I_0(k_1\,a)}) r dr = 2\,\pi\,q\,v_b f_0 \frac{k_0^2}{k_1^2} a^2 (1 - \frac{2}{k_1}\frac{I_1(k_1\,a)}{I_0(k_1\,a)}).$$

Then, the value of the distribution inside the contours defined by (47) is

$$f_0 = \frac{I}{q\,v_b}\,\frac{k_1^2}{2\,\pi k_0^2 a^2}\,\frac{k_1\,I_0(k_1 a)}{k_1 I_0(k_1 a) - 2 I_1(k_1 a)}$$

and ${a'}^2 = b^2(0)$, i.e.

$${a'}^2 = 4\frac{k_0^2}{k_1^2}\left(1 - \frac{1}{I_0(k_1 a)}\right).$$

The Maxwell-Boltzmann distribution. Let us consider the following distribution, which is not compactly supported

$$f(s) = f_1\,\exp(-s/T_\perp).$$

In this case the previous study can not be directly applied, since the radius a is equal to $+\infty$. Then, the space charge density is

$$\begin{aligned} q\,n(r) &= q f_1 \int_{\mathbb{R}^2} \exp(-H_\perp(r, v^2)/T_\perp) dv_r dv_\theta \\ &= 2\,\pi\,T_\perp\,q\,f_1\,\exp(-W(r)/T_\perp) \end{aligned}$$

where the potential energy $W(r)$ is given by (44), and the space charge potential Φ is solution of the nonlinear Poisson equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Phi}{dr}\right) = -2\,\pi\,\frac{q\,f_1\,T_\perp}{\epsilon_0}\exp\left(-\left(\frac{k_0^2}{2}r^2 + \frac{q}{m}(1-\beta^2)\Phi^r\right)/T_\perp\right).$$

where parameters k_0, f_1 and $T_\perp$ have to be defined. Let us rescale the Poisson equation

$$\overline{r} = \frac{2\,k_0^2}{T_\perp}\,r; \quad \overline{\Phi}(\overline{r}) = \frac{q}{m}\frac{(1-\beta^2)}{T_\perp}\Phi(r)$$

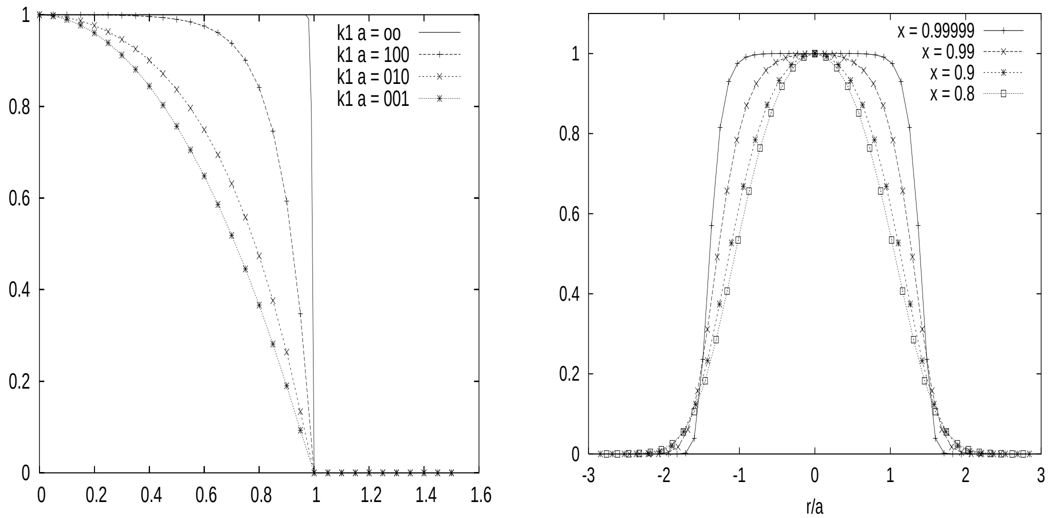
and we get

$$\frac{1}{\overline{r}}\frac{d}{d\overline{r}}\left(\overline{r}\frac{d\overline{\Phi}}{d\overline{r}}\right) = -\alpha_1\,\exp\left(-\left(\overline{r}^2/4 + \overline{\Phi}(\overline{r})\right)\right), \tag{48}$$

where α_1 is

$$\alpha_1 = \frac{\pi}{2}\,\frac{q^2\,f_1}{\epsilon_0\,m}\left(1-\beta^2\right)\left(\frac{T_\perp}{k_0^2}\right)^2.$$

If we assume that $\alpha_1 < 1$, it is then easy to prove from a Banach fixed point argument that the nonlinear Poisson equation (48) has a solution. Finally, let us note that equation (48) cannot be explicitly solved, but numerical approximations can be computed. See Figure 1 for an illustration.



(a) (b)

Figure 1: *The space charge density of the (a) Water-bag distribution with respect to* $k_1\, a$ *(b) Maxwell-Boltzmann distribution with respect to* α_1=0.8,0.9,0.99 *and* 0.99999

6.2 Invariance of the canonical momentum.

In order to reduce the dimension of the problem we use the invariance of the canonical angular momentum

$$P(r, v_\theta) = mrv_\theta$$

Denoting by $I = \frac{P}{m}$ and making the change of variable $(r, v_r, v_\theta) \to (r, v_r, I)$ with $v_\theta = \frac{I}{r}$, we get

$$v_b \frac{\partial f}{\partial z} + v_r \frac{\partial f}{\partial r} + \left(\frac{q}{m}(1-\beta^2)E^s(t,r) + \frac{I^2}{r^3} - \frac{1}{4}\left(\frac{q\,B(z)}{m}\right)^2 r \right) \frac{\partial f}{\partial v_r} = 0.$$

Here the invariant I only appears as a parameter in the equation. It still needs to be discretized but this formulation allows a straightforward parallelization distributing the different values of the invariant over the processors. Then the Vlasov equations corresponding to the values of the invariant are independent. The coupling only occurs through the computation of the charge density. This yields a very efficient parallelization strategy [5].

7 Numerical validation.

In this section, we present numerical results in the four dimensional phase space obtained using a semi-lagrangian method [4, 6].

7.1 Beam focusing via an applied uniform electric field

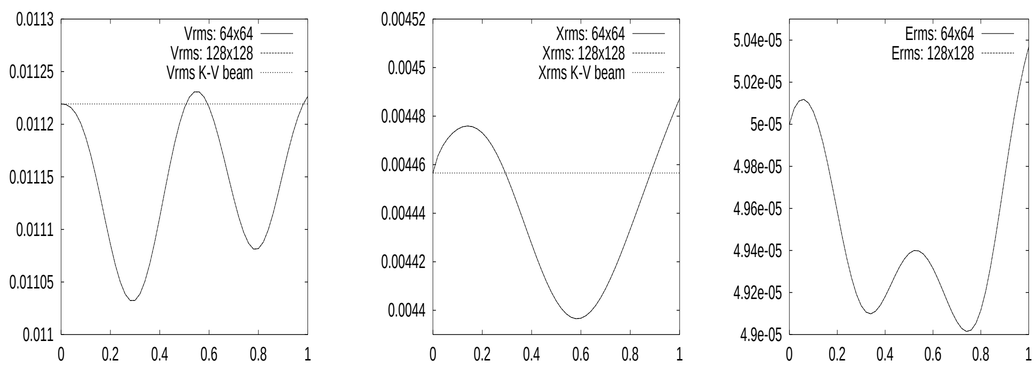
Let us first validate this approach for the case of a semi-Gaussian proton beam in a uniform focusing channel in the four dimensional phase space. The initial value of the distribution function is

$$f_0(x, y, v_x, v_y) = \frac{n_0}{(2\,\pi\, v_{th}^2)\,(\pi\, a^2)}\,\exp\left(-\frac{v_x^2+v_y^2}{2\,v_{th}^2}\right), \quad \text{if } x^2+y^2 \le a^2,$$

and $f_0(x, y, v_x, v_y) = 0$, if $x^2 + y^2 > a^2$ with $a = 8.9\,10^{-2}\,m$, $v_{th} = 6.94\,10^4\,m\,s^{-1}$ and $n_0 = 1.02\,10^{11}$. The applied electric field is given by

$$\vec{E}(x, y) = (E_0\, x, E_0\, y), \quad E_0 = -1.$$

We have chosen the RMS thermal velocity v_{th} such that the equivalent K-V beam is matched. The evolution of the RMS quantities of the semi-Gaussian beam and the equivalent KV beam is displayed in Figure 2 and snapshots of the charge density are displayed in Figure 3. These results agree well with previous results presented in the literature [5, 4]. We notice that the beam at first becomes hollow, then regions of high density propagate to the core of the beam and out again, creating space charge waves. These waves are damped by phase mixing after a few lattice periods.



(a) (b) (c)

Figure 2: *comparison of RMS quantities obtained from an Eulerian Vlasov solver for a semi-Gaussian beam and from a K-V beam: (a) X_{rms}, (b) X'_{rms} and (c) ε_{rms}.*

7.2 Alternating gradient focusing

In this last example, we consider a beam of protons focused with an alternating gradient method: the applied electric field is given by

$$\vec{E}(x, y, z) = (k_0(z), -k_0(z)),$$

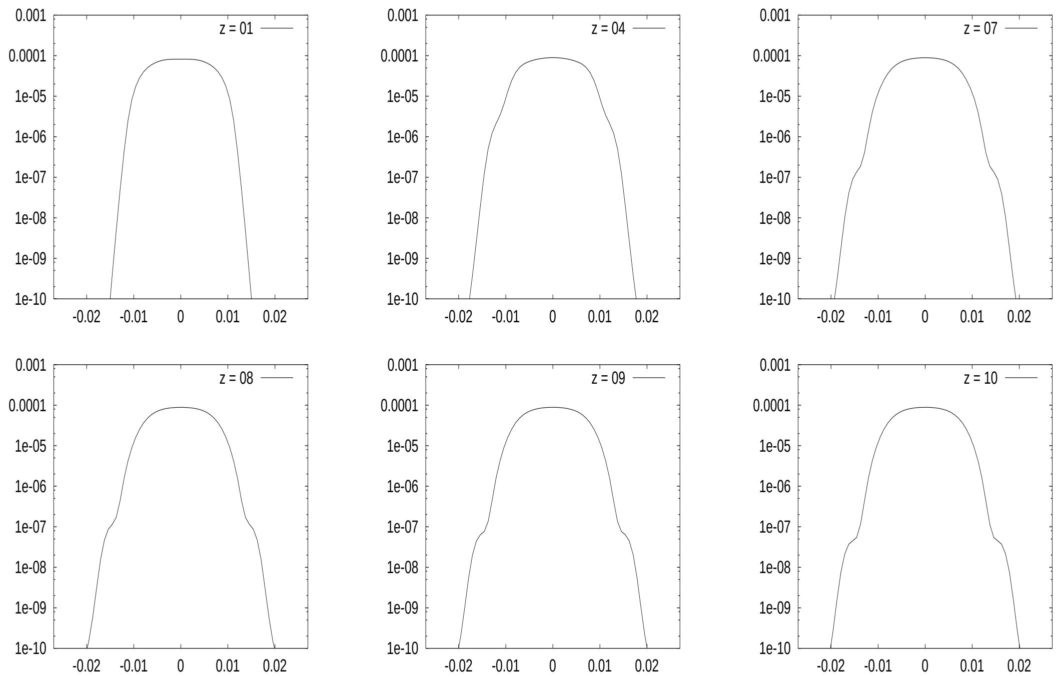


Figure 3: *Evolution of the charge density obtained from an Eulerian Vlasov solver for a Semi-Gaussian beam*

where for $z \in (0,1)$

$$k_0(z) = \begin{cases} +E_0 & \text{if } 0 < z < 1/8, \text{ or } 7/8 < z < 1 \\ 0 & \text{if } 1/8 < z < 3/8, \text{ or } 5/8 < z < 7/8 \\ -E_0 & \text{if } 3/8 < z < 5/8 \end{cases}$$

with $E_0 = 1$. The emittance of the beam is $\epsilon = 2\,10^{-4}\,\pi\, m\, rad$. The initial value of the distribution function is a Maxwell-Boltzmann distribution with $n_0 = \frac{I}{q\, v_b}$, the current $I = 0.1\,A$ and the beam energy is $W = 0.2\,MeV$, so that $u_y = 6.19\,10^{-6}\,m\,s^{-1}$

$$f_0(x,y,v_x,v_y) = \frac{n_0}{4\pi^2\, v_1\, v_2\, b\, a} \exp\left(-\frac{r^2}{2}\right), \quad r^2 = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{v_x}{v_1}\right)^2 + \left(\frac{v_y}{v_2}\right)^2.$$

The evolution of the RMS quantities of the Maxwell-Boltzmann beam and the equivalent KV beam is displayed in Figure 4 and snapshots of the projection on the $r - v_r$ plane of the distribution function are displayed in Figure 5.

8 Conclusion

In this paper we introduce mathematical tools for the numerical simulation of charged particle beams in the paraxial approximation. After recalling the paraxial Vlasov-Poisson system (13)-(14). We introduced three analytical or quasi-analytical steady-state solutions of the model useful for simulation and code validation purposes. One of them, the KV distribution is even more essential, as we were able to show how it can be used to approximately match an arbitrary beam to the external focusing forces. We have thus defined a solid mathematical framework for numerical beam simulations. This framework was applied on some realistic examples using a eulerian solver based on the PFC method on a uniform grid [4, 5, 6]. For larger problems needing more efficiency, we investigate using unstructured grids as well as adaptive mesh refinement.

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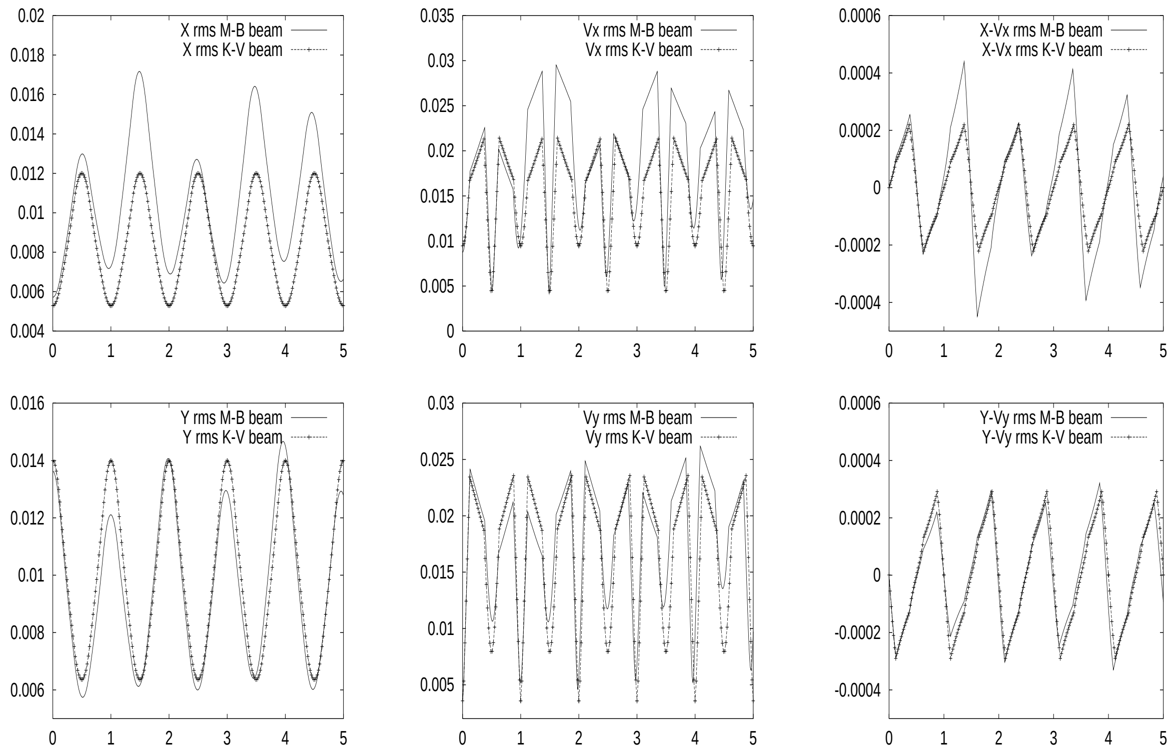

Figure 4: *Evolution of RMS quantities obtained from an Eulerian Vlasov solver for a Maxwell-Boltzmann beam and an equivalent K-V beam*

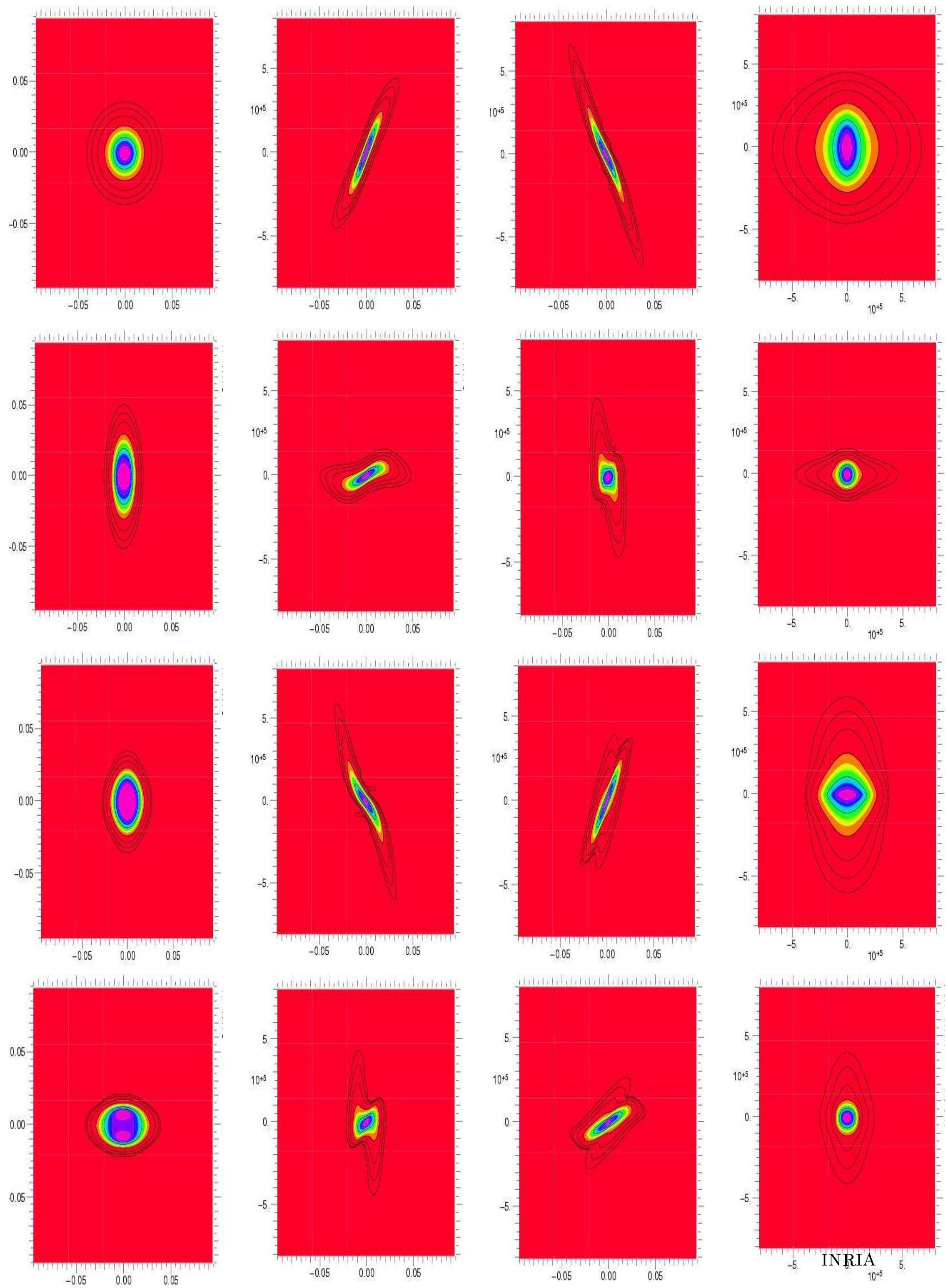

Figure 5: *z evolution of the distribution function $f(z,r,v_r)$ obtained from an Eulerian Vlasov solver for a Maxwell-Boltzmann beam z=0, 1, 2, 4.*

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